

Pretty Good State Transfer on Some NEPS

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Abstract

Let G be a graph with adjacency matrix A . The transition matrix of G relative to A is defined by $H_A(t) := \exp(-itA)$, $t \in \mathbb{R}$. We say that the graph G admits perfect state transfer between the vertices u and v at $\tau \in \mathbb{R}$ if the uv -th entry of $H_A(\tau)$ has unit modulus. Perfect state transfer is a rare phenomena so we consider an approximation called pretty good state transfer. We find that NEPS (Non-complete Extended P-Sum) of the path on three vertices with basis containing tuples with hamming weights of both parities do not exhibit perfect state transfer. But these NEPS admit pretty good state transfer with an additional condition. Further we investigate pretty good state transfer on Cartesian product of graphs and we find that a graph can have PGST from a vertex u to two different vertices v and w .

Keywords: Perfect state transfer, pretty good state transfer, NEPS of graphs.

MSC: 05C50, 15A16

1 Introduction

We consider continuous-time quantum walk relative to the adjacency matrix of a graph. Perfect state transfer in quantum networks was introduced in [4]. Let G be a graph with adjacency matrix A . The transition matrix of G relative to A is defined by

$$H_A(t) := \exp(-itA) = \sum_{n \geq 0} (-i)^n A^n \frac{t^n}{n!}, \quad t \in \mathbb{R}.$$

We say that the graph G exhibits perfect state transfer between the vertices u and v at $\tau \in \mathbb{R}$ if there is $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that $H_A(\tau)e_u = \gamma e_v$. We are mainly interested in finding graphs having perfect state transfer. In [6], we find that Cartesian powers of P_2 which is the path on two vertices, and Cartesian powers of P_3 which is the path on three vertices, admit perfect state transfer. Some other related results are also given in [16]. Further the results have been generalized for the path P_2 in [3, 7]. The question of existence of perfect state transfer in NEPS of P_3 was asked in [17]. In [13], we find that when the hamming weight of each element of the basis of an NEPS of P_3 is of the same parity then an additional condition guarantees perfect state transfer in the NEPS. In this paper, we show that if we allow the basis set to contain elements with hamming weights of both parities, then there is no perfect state transfer in the NEPS. Some more results regarding perfect state transfer can be found in [2, 9, 10, 14, 15].

Perfect state transfer is a rare phenomena. So we consider an approximation to it which is known as pretty good state transfer. We say that a graph G has pretty good state transfer (PGST) from a vertex u to v if, for each $\epsilon > 0$ there exists $t \in \mathbb{R}$ such that

$$|e_u^T H(t) e_v| - 1| < \epsilon,$$

i.e., the modulus of the uv -th entry of the transition matrix comes arbitrary close to 1. There are very few graphs known to have PGST. In [11], Godsil et. al. found that the path P_n admits PGST between the end vertices if and only if $n + 1 = 2^m$, or if $n + 1 = p$ or $2p$ for some odd prime p . A characterization of PGST in double stars have been given in [8]. Some interesting results regarding entries of the transition matrix are also given in [12].

In this paper, we find a class of NEPS of P_3 exhibiting PGST where there is no perfect state transfer. Further we study Cartesian product of graphs and find a way to construct more graphs allowing PGST. As a corollary we obtain a class of NEPS with factor graphs P_2 and P_3 admitting PGST.

Now we define NEPS (Non-complete Extended P-Sum) of n graphs G_1, \dots, G_n with a basis set $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$. The NEPS [5] of G_1, \dots, G_n with basis Ω is denoted by $NEPS(G_1, \dots, G_n; \Omega)$. This NEPS has the vertex set $V(G_1) \times \dots \times V(G_n)$. Two vertices (x_1, \dots, x_n) and (y_1, \dots, y_n) are adjacent in $NEPS(G_1, \dots, G_n; \Omega)$ if and only if there is an n -tuple $(\beta_1, \dots, \beta_n) \in \Omega$ such that $x_i = y_i$ exactly when $\beta_i = 0$ and x_i is adjacent to y_i in G_i exactly when $\beta_i = 1$. Suppose the graphs G_1, \dots, G_n have the adjacency matrices A_1, \dots, A_n , respectively. Then the adjacency matrix of $NEPS(G_1, \dots, G_n; \Omega)$ can be obtained as

$$A_\Omega = \sum_{\beta \in \Omega} A_1^{\beta_1} \otimes \dots \otimes A_n^{\beta_n},$$

where $A \otimes B$ denotes the tensor product of two matrices A and B . Suppose the number of vertices in G_i is n_i . For $i = 1, 2, \dots, n$, let G_i have the eigenvalues $\lambda_{i1}, \dots, \lambda_{in_i}$, not necessarily distinct and suppose the corresponding eigenvectors are $\mathbf{x}_{i1}, \dots, \mathbf{x}_{in_i}$. Then the NEPS with basis Ω has the eigenvalues

$$\Lambda_{j_1 \dots j_n} = \sum_{\beta \in \Omega} \lambda_{1j_1}^{\beta_1} \dots \lambda_{nj_n}^{\beta_n}, \quad j_k = 1, \dots, n_k, \quad k = 1, \dots, n. \quad (1)$$

Here the eigenvector corresponding to $\Lambda_{j_1 \dots j_n}$ is the vector $\mathbf{x}_{1j_1} \otimes \dots \otimes \mathbf{x}_{nj_n}$. See [5] for details.

2 No perfect state transfer in a class of NEPS of P_3

Suppose G is a graph with transition matrix $H_A(t)$ relative to the adjacency matrix A . The graph is said to be periodic at a vertex u if there is $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ and $\tau (\neq 0) \in \mathbb{R}$ so that

$$H_A(\tau)e_u = \gamma e_u.$$

The graph is said to be periodic if it is periodic at all vertices at the same time. In such a case we have $H_A(\tau) = \gamma I$ where I is the identity matrix of appropriate order. It is well known that periodicity is necessary for a graph to exhibit perfect state transfer which follows from the following Lemma.

Lemma 2.1. [10] *If a graph G admits perfect state transfer from a vertex u to another vertex v at time τ then G is periodic at u and v with period 2τ .*

We are dealing with undirected simple graphs so the associated adjacency matrices are symmetric. Assume that A is the adjacency matrix of a graph G . Suppose $\lambda_1, \dots, \lambda_m$ are the

distinct eigenvalues of A and let the projections to the respective eigenspaces be E_1, \dots, E_m . Then the transition matrix of G can be evaluated as

$$H_A(t) = \exp(-itA) = \sum_{r=1}^m \exp(-it\lambda_r) E_r.$$

Suppose the graph G has n vertices. The eigenvalue support of a vector $\mathbf{x} \in \mathbb{R}^n$ is a set containing those eigenvalues λ_r of A for which $E_r \mathbf{x} \neq \mathbf{0}$. Here we include one simple observation. Suppose $E_r = F_1 + \dots + F_k$ where $F_i F_j = \delta_{ij} F_i$ and $F_i^T = F_i$, $1 \leq i, j \leq k$. Here δ_{ij} is the usual Kronecker delta function. Now for any vector \mathbf{x} , we obtain

$$(F_i \mathbf{x})^T E_r \mathbf{x} = (F_i \mathbf{x})^T (F_1 \mathbf{x} + \dots + F_k \mathbf{x}) = \|F_i \mathbf{x}\|^2, \quad 1 \leq i \leq k.$$

Therefore if $F_i \mathbf{x} \neq \mathbf{0}$ for some i then the eigenvalue λ_r belongs to the eigenvalue support of the vector \mathbf{x} . The following theorem characterizes the eigenvalues of a periodic graph.

Theorem 2.2. [10] *Let a graph G be periodic at a vertex u . If $\lambda_k, \lambda_l, \lambda_r, \lambda_s$ are eigenvalues in the eigenvalue support of e_u and $\lambda_r \neq \lambda_s$ then*

$$\frac{\lambda_k - \lambda_l}{\lambda_r - \lambda_s} \in \mathbb{Q}. \quad (2)$$

Assume that $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ which contains tuples with hamming weights of both parities. Now we find that NEPS of P_3 with basis Ω does not exhibit perfect state transfer. We denote $s(\beta)$ to be the hamming weight of a tuple β .

Theorem 2.3. *Suppose Ω_e and Ω_o are both nonempty subsets of $\mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ such that the hamming weight of each tuple in Ω_e and Ω_o are even and odd, respectively. If $\Omega = \Omega_e \cup \Omega_o$ then $NEPS(P_3, \dots, P_3; \Omega)$ does not exhibit perfect state transfer.*

Proof. The graph P_3 has the eigenvalues $-\sqrt{2}$, 0 , $\sqrt{2}$ and the corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Therefore the eigenvectors of $NEPS(P_3, \dots, P_3; \Omega)$ are

$$\mathbf{x}_{j_1} \otimes \dots \otimes \mathbf{x}_{j_n}, \text{ where } j_1, \dots, j_n \in \{1, 2, 3\}.$$

Notice that, if $j_1, \dots, j_n \in \{1, 3\}$ then all columns of $(\mathbf{x}_{j_1} \otimes \dots \otimes \mathbf{x}_{j_n})(\mathbf{x}_{j_1} \otimes \dots \otimes \mathbf{x}_{j_n})^T$ are nonzero. Hence, using the observation preceding Theorem 2.2, the eigenvalue support of all standard unit vectors in \mathbb{R}^{3^n} contains the eigenvalue corresponding to the vector $\mathbf{x}_{j_1} \otimes \dots \otimes \mathbf{x}_{j_n}$.

Observe that the hamming weight $s(\beta)$ is even or odd according as β is in Ω_e or Ω_o . Using equation (1), the eigenvalue corresponding to $\mathbf{x}_1 \otimes \mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_1$ can be calculated as

$$\begin{aligned} \sum_{\beta \in \Omega} (-\sqrt{2})^{s(\beta)} &= \sum_{\beta \in \Omega_e} (-\sqrt{2})^{s(\beta)} + \sum_{\beta \in \Omega_o} (-\sqrt{2})^{s(\beta)} \\ &= \sum_{\beta \in \Omega_e} (\sqrt{2})^{s(\beta)} - \sum_{\beta \in \Omega_o} (\sqrt{2})^{s(\beta)} \\ &= a - b\sqrt{2}, \text{ where } a \text{ and } b \text{ are positive integers.} \end{aligned}$$

Similarly the eigenvalue corresponding to the vector $\mathbf{x}_3 \otimes \mathbf{x}_3 \otimes \dots \otimes \mathbf{x}_3$ can be obtained in terms of a and b as

$$\begin{aligned} \sum_{\beta \in \Omega} (\sqrt{2})^{s(\beta)} &= \sum_{\beta \in \Omega_e} (\sqrt{2})^{s(\beta)} + \sum_{\beta \in \Omega_o} (\sqrt{2})^{s(\beta)} \\ &= a + b\sqrt{2}. \end{aligned}$$

For $1 \leq j \leq n$ consider $\Omega_e^j = \{\beta \in \Omega_e : \beta_j = 1\}$. Since Ω_e is a non-empty subset of $\mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ there is at least one j for which Ω_e^j is also non-empty. Without loss of generality let $j = 1$ and hence the integral part of the eigenvalue corresponding to $\mathbf{x}_1 \otimes \mathbf{x}_3 \otimes \dots \otimes \mathbf{x}_3$ is

$$\sum_{\beta \in \Omega_e^1} (-\sqrt{2}) (\sqrt{2})^{s(\beta)-1} + \sum_{\beta \in \Omega_e \setminus \Omega_e^1} (\sqrt{2})^{s(\beta)},$$

which is clearly not equal to the integer a . Assume that the eigenvalue corresponding to the vector $\mathbf{x}_1 \otimes \mathbf{x}_3 \otimes \dots \otimes \mathbf{x}_3$ is $c + d\sqrt{2}$, for some integers $c (\neq a)$ and d .

Now if $NEPS(P_3, \dots, P_3; \Omega)$ exhibits perfect state transfer then by Lemma 2.1 the graph is periodic at some vertex, say u . Hence by Theorem 2.2 the eigenvalues in the eigenvalue support of the characteristic vector of u must satisfy the ratio condition (2). The eigenvalues $a + b\sqrt{2}$, $a - b\sqrt{2}$ and $c + d\sqrt{2}$ lies in the eigenvalue support of e_u . Note that

$$\begin{aligned} \frac{(a + b\sqrt{2}) - (a - b\sqrt{2})}{(a + b\sqrt{2}) - (c + d\sqrt{2})} &= \frac{2b\sqrt{2}}{(a - c) + (b - d)\sqrt{2}} \\ &= \frac{2b\sqrt{2} [(a - c) - (b - d)\sqrt{2}]}{(a - c)^2 - 2(b - d)^2} \notin \mathbb{Q}. \end{aligned}$$

Therefore $NEPS(P_3, \dots, P_3; \Omega)$ is not periodic at all vertices and hence the graph does not exhibit perfect state transfer. \square

The above result gives a partial characterization of perfect state transfer in the class of all NEPS of P_3 . In the following section we investigate PGST on NEPS of P_3 with basis Ω containing tuples with hamming weights of both parities.

3 Pretty good state transfer on NEPS of P_3

Before discussing PGST on NEPS we mention some previously known results. The following result shows that transition matrix of an NEPS can be realized as a product of transition matrices of some of its spanning subgraphs.

Proposition 3.1. [13] *Let G_1, \dots, G_n be n graphs and consider $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$. For $\beta \in \Omega$, let $H_\beta(t)$ be the transition matrix of NEPS $(G_1, \dots, G_n; \{\beta\})$. Then NEPS $(G_1, \dots, G_n; \Omega)$ has the transition matrix*

$$H_\Omega(t) = \prod_{\beta \in \Omega} H_\beta(t).$$

Let us denote $\tau_k = \frac{\pi}{(\sqrt{2})^k}$ for $k \in \mathbb{N}$. Then we have the following lemma.

Lemma 3.2. [13] *Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ and suppose $H_\beta(t)$ is the transition matrix of NEPS $(P_3, \dots, P_3; \{\beta\})$. If the hamming weight of β is k then $H_\beta(-\tau_k) = H_\beta(\tau_k)$.*

Now we find the transition matrix of an NEPS of P_3 at a specific time depending on the basis of the NEPS.

Theorem 3.3. [13] *Consider $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ such that for all $\beta \in \Omega$, the number $s(\beta)$ is even (or odd). Let $k = \min_{\beta \in \Omega} s(\beta)$ and $\Omega^* = \{\beta \in \Omega : s(\beta) = k\}$. If the transition matrices of NEPS of P_3 corresponding to Ω and Ω^* are $H_\Omega(t)$ and $H_{\Omega^*}(t)$, respectively, then $H_\Omega(\tau_k) = H_{\Omega^*}(\tau_k)$.*

In the following result we find that an NEPS with basis containing tuples having hamming weights odd (or even) is essentially periodic.

Theorem 3.4. *Suppose $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ satisfies the conditions of Theorem 3.3. Then the graph NEPS $(P_3, \dots, P_3; \Omega)$ is periodic at $2\tau_k$.*

Proof. Using Lemma 3.2, for $\beta \in \Omega^*$, we find that $H_\beta(-\tau_k) = H_\beta(\tau_k)$. Further, Proposition 3.1

implies that

$$\begin{aligned}
H_{\Omega^*}(-\tau_k) &= \prod_{\beta \in \Omega^*} H_{\beta}(-\tau_k) \\
&= \prod_{\beta \in \Omega^*} H_{\beta}(\tau_k) \\
&= H_{\Omega^*}(\tau_k).
\end{aligned}$$

Therefore we obtain $H_{\Omega^*}(2\tau_k) = I$. Now Theorem 3.3 gives

$$H_{\Omega}(2\tau_k) = (H_{\Omega}(\tau_k))^2 = (H_{\Omega^*}(\tau_k))^2 = H_{\Omega^*}(2\tau_k) = I.$$

Hence $NEPS(P_3, \dots, P_3; \Omega)$ is periodic at $2\tau_k$. \square

Note that if k is even then $\tau_k = \frac{\pi}{(\sqrt{2})^k} = \frac{\pi}{2^{k/2}}$. Recall that if a graph is periodic at τ then it is periodic at $2^r \tau$ for all non-negative integer r . As an implication, we have the following obvious corollary which will be used to find PGST in some NEPS of P_3 .

Corollary 3.5. *Let $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ be such that the hamming weight of β for each $\beta \in \Omega$ is even. Then the NEPS of P_3 corresponding to Ω is periodic at π .*

Consider an NEPS with basis Ω . Let $M(\Omega)$ be the matrix formed by writing the vectors in Ω as its rows. We denote the rank of $M(\Omega)$ over \mathbb{Z}_2 by $r(\Omega)$.

Theorem 3.6. *[18] Let B_1, \dots, B_n be connected bipartite graphs. Then $NEPS(B_1, \dots, B_n; \Omega)$ is connected if and only if the rank $r(\Omega) = n$.*

The following result gives a sufficient condition for an NEPS of P_3 to be connected and exhibit perfect state transfer. Since we are interested in connected graphs we impose the condition $r(\Omega) = n$ in the following result only to ensure that the graphs involved are connected.

Theorem 3.7. *[13] Consider $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ such that $r(\Omega) = n$. Also, for $\beta \in \Omega$, assume that $s(\beta)$ is even (or odd). Suppose that $k = \min_{\beta \in \Omega} s(\beta)$ and $\Omega^* = \{\beta \in \Omega : s(\beta) = k\}$. If $\sum_{\beta \in \Omega^*} \beta \neq \mathbf{0}$ in \mathbb{Z}_2^n then $NEPS(P_3, \dots, P_3; \Omega)$ allows perfect state transfer at time $\frac{\pi}{(\sqrt{2})^k}$.*

Consider the following approximation theorem by Kronecker which plays a crucial role in finding PGST in the later results.

Theorem 3.8 (Kronecker's approximation theorem). [1] *Let θ be an irrational number and suppose α is a real number. For every $\delta > 0$ there are integers p and q such that*

$$|p\theta - q - \alpha| < \delta.$$

Suppose Ω_e and Ω_o are non-empty subsets of $\mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ containing tuples with hamming weights even and odd, respectively.

Theorem 3.9. *Let $\Omega = \Omega_e \cup \Omega_o \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ be such that both Ω_e, Ω_o are non-empty and $r(\Omega) = n$. Assume that $k = \min_{\beta \in \Omega_e} s(\beta)$ and $l = \min_{\beta \in \Omega_o} s(\beta)$ with $\Omega_e^* = \{\beta \in \Omega_e : s(\beta) = k\}$ and $\Omega_o^* = \{\beta \in \Omega_o : s(\beta) = l\}$. Then pretty good state transfer occurs in $NEPS(P_3, \dots, P_3; \Omega)$ if any one of the following conditions holds:*

1. $\sum_{\beta \in \Omega_o^*} \beta \neq \mathbf{0}$ in \mathbb{Z}_2^n , or
2. $\sum_{\beta \in \Omega_e^*} \beta \neq \mathbf{0}$ in \mathbb{Z}_2^n .

Proof. Proposition 3.1 implies that $H_\Omega(t) = H_{\Omega_e}(t)H_{\Omega_o}(t)$. As k is even, by Corollary 3.5, we find that $NEPS(P_3, \dots, P_3; \Omega_e)$ is periodic at π and hence for all integer q we have $H_{\Omega_e}(q\pi) = I$. Let us denote $\tau = \frac{\pi}{(\sqrt{2})^l}$ and $\eta = \frac{\pi}{(\sqrt{2})^k}$.

Case I: Assume that $\sum_{\beta \in \Omega_o^*} \beta \neq \mathbf{0}$ in \mathbb{Z}_2^n . By Theorem 3.7, the graph $NEPS(P_3, \dots, P_3; \Omega_o)$ admits perfect state transfer at τ , say, between the pair of vertices u, v , i.e., $|e_u^T H_{\Omega_o}(\tau) e_v| = 1$. Let us consider

$$f(t) = |e_u^T H_{\Omega_o}(t) e_v|,$$

which is necessarily a continuous function. By Theorem 3.4, the function $f(t)$ is also periodic with period 2τ and therefore $f(t)$ is uniformly continuous on \mathbb{R} . So for $\epsilon > 0$, there exist $\delta > 0$ such that $|t - t'| < \delta$ implies $|f(t) - f(t')| < \epsilon$. Consider $\alpha = \frac{1}{(\sqrt{2})^l}$ and $\theta = 2\alpha$. Since l is an odd number the chosen number θ is indeed an irrational number. By Theorem 3.8, for $\delta > 0$ there exists integers p and q such that $|p\theta - q - \alpha| < \frac{\delta}{\pi}$, i.e., $|(2p-1)\alpha - q| < \frac{\delta}{\pi}$. Now $|(2p-1)\tau - q\pi| < \delta$ implies that

$$|f((2p-1)\tau) - f(q\pi)| < \epsilon.$$

Now notice that $f((2p-1)\tau) = f(\tau) = 1$ and hence $|f(q\pi) - 1| < \epsilon$. Therefore we obtain

$$H_\Omega(q\pi) = H_{\Omega_e}(q\pi)H_{\Omega_o}(q\pi) = H_{\Omega_o}(q\pi),$$

which in turn implies that $f(q\pi) = |e_u^T H_\Omega(q\pi) e_v|$. Thus for $\epsilon > 0$ there exists $q\pi \in \mathbb{R}$ such that

$$||e_u^T H_\Omega(q\pi) e_v| - 1| < \epsilon.$$

So $NEPS(P_3, \dots, P_3; \Omega)$ has PGST between the pair of vertices u and v .

Case II: Suppose $\sum_{\beta \in \Omega_e^*} \beta \neq \mathbf{0}$ in \mathbb{Z}_2^n . By Theorem 3.7, the graph $NEPS(P_3, \dots, P_3; \Omega_e)$ exhibits perfect state transfer at η , say, between the pair of vertices u and v . Consider

$$g(t) = |e_u^T H_{\Omega_e}(t) e_v|.$$

By the same argument as in Case I, the function $g(t)$ is uniformly continuous. Therefore for $\epsilon > 0$, there exist $\delta > 0$ so that $|t - t'| < \delta$ implies $|g(t) - g(t')| < \epsilon$. Notice that for every integer q , we have $g(q\pi + \eta) = g(\eta) = 1$. Consider $\theta = \frac{2}{(\sqrt{2})^l}$, which is an irrational number as l is an odd natural number. Also let $\alpha = \frac{1}{(\sqrt{2})^k}$. Then by Theorem 3.8, for $\delta > 0$ there exists integers p and q such that $|p\theta - q - \alpha| < \frac{\delta}{\pi}$. Now $|2p\tau - (q\pi + \eta)| < \delta$ implies that

$$|g(2p\tau) - g(q\pi + \eta)| < \epsilon, \text{ i.e., } |g(2p\tau) - 1| < \epsilon.$$

Finally we have

$$H_\Omega(2p\tau) = H_{\Omega_e}(2p\tau) H_{\Omega_o}(2p\tau) = H_{\Omega_e}(2p\tau), \text{ as } H_{\Omega_o}(2p\tau) = I.$$

So for each $\epsilon > 0$ there exists $2p\tau \in \mathbb{R}$ so that $||e_u^T H_\Omega(2p\tau) e_v| - 1| < \epsilon$. Hence PGST occurs between the pair of vertices u and v in $NEPS(P_3, \dots, P_3; \Omega)$. \square

Thus we find an infinite class of graphs allowing PGST. In the following section we find some other graphs exhibiting PGST.

4 Pretty good state transfer on Cartesian products

The Cartesian product of two graphs G_1 and G_2 with vertex sets V_1 and V_2 is the graph $G_1 \square G_2$, with vertex set $V_1 \times V_2$. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \square G_2$ if and only if either u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$, or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 . The transition matrix of a Cartesian product of two graphs is given by the following result.

Lemma 4.1. [6] *Let G_1 and G_2 be two graphs and suppose $G_1 \square G_2$ is the Cartesian product of G_1 and G_2 . If G_1 and G_2 have the transition matrices $H_{G_1}(t)$ and $H_{G_2}(t)$, respectively, then the transition matrix of $G_1 \square G_2$ is*

$$H_{G_1 \square G_2}(t) = H_{G_1}(t) \otimes H_{G_2}(t).$$

Now we investigate PGST on Cartesian product of two graphs.

Theorem 4.2. *Let G_1 and G_2 be two graphs so that G_1 is periodic at a vertex at τ and G_2 exhibits perfect state transfer at η . If τ and η are independent over the rational numbers then $G_1 \square G_2$ admits pretty good state transfer.*

Proof. Since τ and η are independent over \mathbb{Q} , by Kronecker's approximation theorem the set $\{m\tau - 2n\eta : m, n \in \mathbb{Z}\}$ is dense in \mathbb{R} . Hence for $\delta > 0$, there exists $m, n \in \mathbb{Z}$ so that

$$|m\tau - (2n + 1)\eta| < \delta.$$

Suppose G_1 is periodic at the vertex u at τ , i.e., $|e_u^T H_{G_1}(m\tau) e_u| = 1$ for all integer m . Also assume that G_2 exhibits perfect state transfer at η between the pair of vertices v and w . Consider $f(t) = |e_v^T H_{G_2}(t) e_w|$. For each integer n we have

$$\begin{aligned} f((2n + 1)\eta) &= |e_v^T H_{G_2}((2n + 1)\eta) e_w| \\ &= |e_v^T H_{G_2}(2n\eta) H_{G_2}(\eta) e_w| \\ &= |e_v^T H_{G_2}(2n\eta) \gamma e_v|, \text{ as } H_{G_2}(\eta) e_w = \gamma e_v, \text{ for some } \gamma \in \mathbb{C} \text{ with } |\gamma| = 1 \\ &= |e_v^T H_{G_2}(2n\eta) e_v|. \end{aligned}$$

By Lemma 2.1, the graph G_2 is periodic at the vertex v at 2η and therefore

$$1 = |e_v^T H_{G_2}(2n\eta) e_v| = f((2n + 1)\eta).$$

Now $f(t)$ is uniformly continuous and therefore for $\epsilon > 0$ there is $m, n \in \mathbb{Z}$ so that

$$|f(m\tau) - f((2n + 1)\eta)| < \epsilon, \text{ i.e., } |f(m\tau) - 1| < \epsilon.$$

Since $H_{G_1 \square G_2}(t) = H_{G_1}(t) \otimes H_{G_2}(t)$, using properties of tensor product, we obtain

$$\begin{aligned} |(e_u \otimes e_v)^T (H_{G_1 \square G_2}(m\tau)) (e_u \otimes e_w)| &= |(e_u \otimes e_v)^T (H_{G_1}(m\tau) \otimes H_{G_2}(m\tau)) (e_u \otimes e_w)| \\ &= |(e_u^T H_{G_1}(m\tau) e_u) \otimes (e_v^T H_{G_2}(m\tau) e_w)| \\ &= f(m\tau). \end{aligned}$$

Hence $G_1 \square G_2$ has PGST between the pair of vertices (u, v) and (u, w) . \square

It is interesting to see that a graph can have PGST from a vertex u to two different vertices v and w . However, it is well known that if $v \neq w$ then there cannot be perfect state transfer from u to v and also from u to w . Consider the following example.

Example 4.1. *The path P_2 with vertices, say u, v , exhibits perfect state transfer between the pair of vertices u and v at $\frac{\pi}{2}$. It is also well known that P_2 is periodic at π . On the other hand the path P_3 with vertices, say 1, 2 and 3, admits perfect state transfer between the pair of vertices 1 and 3 at $\frac{\pi}{\sqrt{2}}$. The path P_3 is also periodic at $\frac{2\pi}{\sqrt{2}}$. Hence by Theorem 4.2, the Cartesian product $P_2 \square P_3$ allows PGST between the pair of vertices $(u, 1)$ and $(u, 3)$. Again we find that $P_3 \square P_2$ admits PGST between the vertices $(1, u)$ and $(1, v)$. Since $P_3 \square P_2 \cong P_2 \square P_3$, we see that $P_2 \square P_3$ also exhibits PGST between the pair of vertices $(u, 1)$ and $(v, 1)$.*

Now using Theorem 4.2, we find that some of the NEPS with factor graphs P_2 and P_3 which can be realized as a Cartesian product of an NEPS of P_2 and an NEPS of P_3 , exhibits PGST.

Corollary 4.3. *Let $\Omega \subset \mathbb{Z}_2^m \setminus \{\mathbf{0}\}$ and $\Omega' \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$. Suppose the hamming weight of each tuple in Ω is odd, $k = \min_{\beta \in \Omega} s(\beta)$ and $\Omega^* = \{\beta \in \Omega : s(\beta) = k\}$. If $\sum_{\beta \in \Omega^*} \beta \neq \mathbf{0}$ or $\sum_{\beta \in \Omega'} \beta \neq \mathbf{0}$ then the Cartesian product of NEPS $(P_3, \dots, P_3; \Omega)$ and NEPS $(P_2, \dots, P_2; \Omega')$ exhibits PGST.*

Proof. Assume that $\sum_{\beta \in \Omega^*} \beta \neq \mathbf{0}$ in \mathbb{Z}_2^m . Then by Theorem 3.7, NEPS $(P_3, \dots, P_3; \Omega)$ admits perfect state transfer at $\frac{\pi}{(\sqrt{2})^k}$. Further, Lemma 7.2 in [9] implies that NEPS $(P_2, \dots, P_2; \Omega')$ is periodic at π . Since k is odd, the numbers π and $\frac{\pi}{(\sqrt{2})^k}$ are independent over \mathbb{Q} . Therefore, by applying Theorem 3.9, we see that the Cartesian product of NEPS $(P_2, \dots, P_2; \Omega')$ and NEPS $(P_3, \dots, P_3; \Omega)$ exhibits PGST. Hence we have PGST in the Cartesian product of NEPS $(P_3, \dots, P_3; \Omega)$ and NEPS $(P_2, \dots, P_2; \Omega')$.

Suppose $\sum_{\beta \in \Omega'} \beta \neq \mathbf{0}$ in \mathbb{Z}_2^n . Then by Lemma 7.2 in [9], the graph NEPS $(P_2, \dots, P_2; \Omega')$ admits perfect state transfer at $\frac{\pi}{2}$. Also, by Theorem 3.4, we find that NEPS $(P_3, \dots, P_3; \Omega)$ is periodic at $\frac{2\pi}{(\sqrt{2})^k}$. Hence, by Theorem 3.9, the Cartesian product of NEPS $(P_3, \dots, P_3; \Omega)$ and NEPS $(P_2, \dots, P_2; \Omega')$ exhibits PGST. \square

Conclusions

In [13], we have seen that there are many NEPS of P_3 allowing perfect state transfer. Here we considered the case when the basis of an NEPS of P_3 contains tuples with hamming weights of both parities. We found that in such cases there is no perfect state transfer. Then we looked for PGST in that class of graphs. We found that many of them admit PGST.

The only class of NEPS of P_3 where we still do not know whether there is perfect state transfer are those NEPS in Theorem 3.7 where $\sum_{\beta \in \Omega^*} \beta = \mathbf{0}$ in \mathbb{Z}_2^n . By Theorem 3.4, we find that these graphs are essentially periodic. Therefore, if a graph in that class has PGST then it must also exhibit perfect state transfer.

Finally, in Theorem 4.2, we found a general result on PGST on Cartesian product of graphs. Using that we have seen that there are NEPS with factor graphs P_2 and P_3 exhibiting PGST. Following this we can construct many other type of graphs allowing PGST.

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